Sec 5.3 (Part 2)

$y'' + y = 0$ const. coeff, homog.

$\Rightarrow$ use char. eqn $r^2 + 1 = 0$

$\Rightarrow r^2 = -1 \Longrightarrow r = \pm\sqrt{-1} = \pm i$

can also factor

$$r^2 + 1 = 0 \longrightarrow (r-i)(r+i) = 0$$

- So gen solutions are all

$$y = c_1 \underbrace{e^{ix}} + c_2 \underbrace{e^{-ix}}$$

2^{nd} order $\rightarrow$ 2 lin. indep func.

Hmm... $y'' + y = 0 \Rightarrow y'' = -y$

$y = \cos x \rightarrow y' = -\sin x \rightarrow y'' = -\cos x$

$y = \sin x \rightarrow y' = \cos x \rightarrow y'' = -\sin x$

so $\{\cos x, \sin x\}$ are two lin indep solutions... $\Rightarrow$ all solutions are lin comb.

$$y = c_1 \cos x + c_2 \sin x \; ?$$

$\cos x = c_1 e^{ix} + c_2 e^{-ix}$?

$\sin x = \tilde{c}_1 e^{ix} + \tilde{c}_2 e^{-ix}$?

vice versa?

☆ Exponential series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

try

$$e^{(ix)} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \cdots$$

check : $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $i^5 = i$, $\cdots$

(cycle repeats)

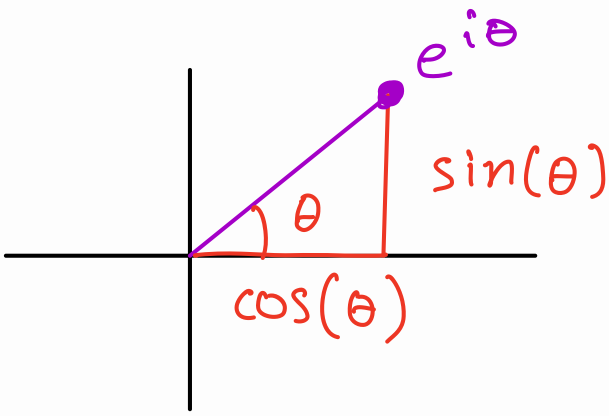
$$\Longrightarrow e^{ix} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + i\frac{x^5}{120} + \cdots$$

$$\underbrace{\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)}_{\text{Taylor series for } \cos x} + i\left(\underbrace{x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots}_{\text{series for } \sin x}\right)$$

$$\boxed{e^{ix} = \cos x + i\sin x} \quad (1)$$

"Euler's formula"

- $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$



- $$\boxed{e^{-ix} = \cos(x) - i\sin(x)} \quad (2)$$

$\frac{(1)+(2)}{2} \Rightarrow$ $\frac{(1)-(2)}{2i}$ note

$$\boxed{\begin{aligned} \cos x &= \frac{e^{ix}+e^{-ix}}{2} \\ \sin x &= \frac{e^{ix}-e^{-ix}}{2i} \end{aligned}}$$

Prefer non-complex solutions for mech. problems.

Ex: $y'' + 25y = 0$

ch. eq.: $r^2 + 25 = 0 \Rightarrow r = \pm 5i$

so gen solution

$$y = \underline{c_1 e^{5ix}} + \underline{c_2 e^{-5ix}}$$

prefer to convert-to/use cos, sin basis

Euler's formula $\Rightarrow$ $y = c_1(\underline{\cos(5x) + i\sin(5x)})$

$$+ c_2(\underline{\cos(5x) - i\sin(5x)})$$

$$= \underbrace{(c_1+c_2)}_{\text{unknown const "A"}}\cos(5x) + \underbrace{i(c_1-c_2)}_{\text{unknown const "B"}}\sin(5x)$$

gen solution $\boxed{y = A\cos(5x) + B\sin(5x)}$

Notes ☆

- complex roots always come in <u>conjugate pairs</u> (<u>Ex:</u> $0 \pm 5i$, $3 \pm 2i$, $\pi \pm 7i$, ...)
- conjugate pairs of roots produce pairs of solutions:

$$\boxed{\text{root pair } \quad r = a \pm bi}$$

$\longrightarrow$ solution pair $\quad c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}$

$$= e^{ax}\left(\underbrace{c_1 e^{i(bx)} + c_2 e^{i(-bx)}}_{\text{convert} \ldots}\right)$$

$$= e^{ax}\left(A\cos(bx) + B\sin(bx)\right)$$

$$= \boxed{Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)}$$

("$a \pm bi$ piece")

Ex: $\quad y^{(3)} - 3y'' + 25y' - 75y = 0$

ch. eqn $\quad r^3 - 3r^2 + 25r - 75 = 0$

$\times 3 \qquad \times 3$

$$r^2(r-3) + 25(r-3) = 0$$

$$\underbrace{(r-3)}_{\substack{r=3 \\ \text{mult } 1}}\underbrace{(r^2+25)}_{\substack{r = \pm 5i \\ \text{mult } 1}} = 0$$

(alt: $r = 0 \pm 5i$)

So gen solution is

$$y = (3 \text{ piece}) + (\pm 5i \text{ piece})$$

$$= c_1 e^{3x} + \left(c_2 \cos(5x) + c_3 \sin(5x)\right)$$

(we skipped "conversion")

★ for $\cos(5x)$, $\sin(5x)$ we basically took real and imaginary parts of solution

$$e^{i(5x)} = \boxed{\cos(5x)} + i\boxed{\sin(5x)}$$

Read "order 4 example"

Order 4 example

Ex : $2y^{(4)} + 7y'' + 3 = 0$

ch. eq : $2r^4 + 7r^2 + 3 = 0$

(only powers of r^2 present)

trick : $s = r^2$

ch. eq $\rightarrow$ $2s^2 + 7s + 3 = 0$

quad form $\Longrightarrow$ $s = \dfrac{-7 \pm \sqrt{49-24}}{4}$

$$s = \frac{-7 \pm \sqrt{25}}{4} = \frac{-7 \pm 5}{4} \Longrightarrow s = -3 \text{ or } s = -\frac{1}{2}$$

$$\begin{array}{l} s=-3 \rightarrow r^2 = -3 \Rightarrow \\ s = -1/2 \rightarrow r^2 = -1/2 \Rightarrow \end{array} \boxed{\begin{array}{l} r = \pm\sqrt{3}\, i \\ r = \pm \frac{1}{\sqrt{2}}\, i \end{array}} \Big\} \text{ two conjugate pairs}$$

so general solution is

$$y = \left(\underline{\pm\sqrt{3}\, i \text{ piece}}\right) + \left(\underline{\pm \tfrac{1}{\sqrt{2}}\, i \text{ piece}}\right)$$

$$= \left(\underline{c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x)}\right)$$

$$+ \left(\underline{c_3 \cos\left(\tfrac{1}{\sqrt{2}}x\right) + c_4 \sin\left(\tfrac{1}{\sqrt{2}}x\right)}\right)$$